Chapter 16
Solving Differential Equations

This chapter describes how to solve both ordinary and partial differential equations having real-valued solutions. Mathcad Standard comes with the *rkfixed* function, a general-purpose Runge-Kutta solver that can be used on *n*th order differential equations with initial conditions or on systems of differential equations. Mathcad Professional includes a variety of additional, more specialized functions for solving differential equations. Some of these exploit properties of the differential equation to improve speed and accuracy. Others are useful when you intend to plot the solution rather than simply evaluate it at an endpoint.

The following sections make up this chapter:

Solving ordinary differential equations

Using the *rkfixed* function to solve an *n*th order ordinary differential equation with initial conditions. This section is a prerequisite for all other sections in this chapter.

Systems of differential equations

How to adapt the *rkfixed* function to solve systems of differential equations with initial conditions.

Pro **Specialized differential equation solvers**

A description of additional differential equation solving functions and when you may want to use them.

Pro **Boundary value problems**

How to solve boundary value problems involving multivariate functions.

Solving ordinary differential equations

In a differential equation, you solve for an unknown function rather than just a number. For ordinary differential equations, the unknown function is a function of one variable. Partial differential equations are differential equations in which the unknown is a function of two or more variables.

Mathcad has a variety of functions for returning the solution to an ordinary differential equation. Each of these functions solves differential equations numerically. You'll always get back a matrix containing the values of the function evaluated over a set of points. These functions differ in the particular algorithm each uses for solving differential equations. Despite these differences however, each of these functions requires you to specify at least three things:

- The initial conditions.
- A range of points over which you want the solution to be evaluated.
- The differential equation itself, written in the particular form discussed in this chapter.

This section shows how to solve a single ordinary differential equation using the function *rkfixed*. It begins with an example of how to solve a simple first order differential equation and then proceeds to show how to solve higher order differential equations.

First order differential equations

A first order differential equation is one in which the highest order derivative of the unknown function is the first derivative. Figure 16-1 shows an example of how to solve the relatively simple differential equation:

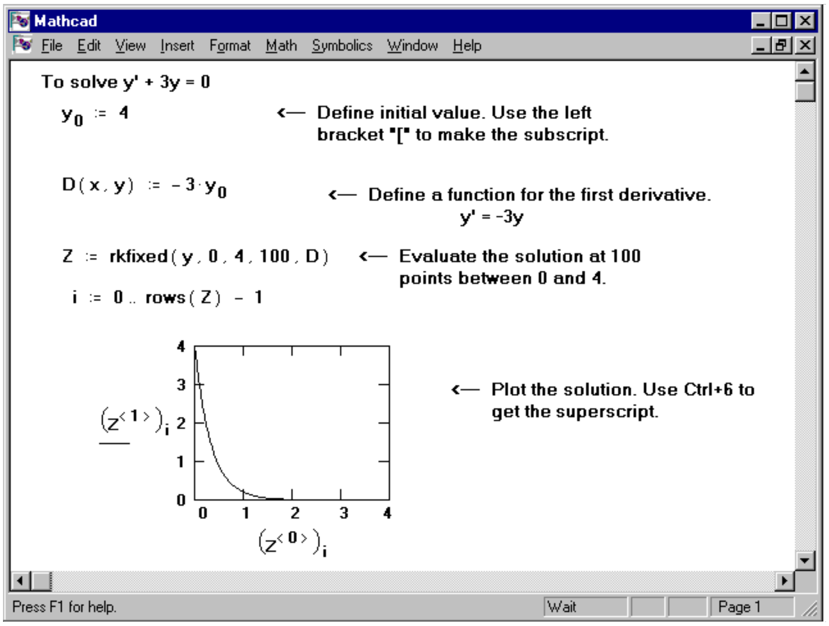
$$\frac{dy}{dx} + 3 \cdot y = 0$$

subject to the initial condition:

$$y(0) = 4$$

The function *rkfixed* in Figure 16-1 uses the fourth order Runge-Kutta method to return a two-column matrix in which:

- The left-hand column contains the points at which the solution to the differential equation is evaluated.
- The right-hand column contains the corresponding values of the solution.


Mathcad
File Edit View Insert Format Math Symbolics Window Help
Press F1 for help.
Wait
Page 1

To solve y' + 3y = 0

$y_0 := 4$ <— Define initial value. Use the left bracket "[" to make the subscript.

$D(x, y) := -3 \cdot y_0$ <— Define a function for the first derivative. y' = -3y

$Z := \text{rkfixed}(y, 0, 4, 100, D)$ <— Evaluate the solution at 100 points between 0 and 4.

$i := 0 .. \text{rows}(Z) - 1$

<— Plot the solution. Use Ctrl+6 to get the superscript.

$(Z^{\langle 1 \rangle})_i$
$(Z^{\langle 0 \rangle})_i$
0
1
2
3
4

Figure 16-1: Solving a first order differential equation.

The arguments to the *rkfixed* function are:

rkfixed(**y**, *x1*, *x2*, *npoints*, **D**)

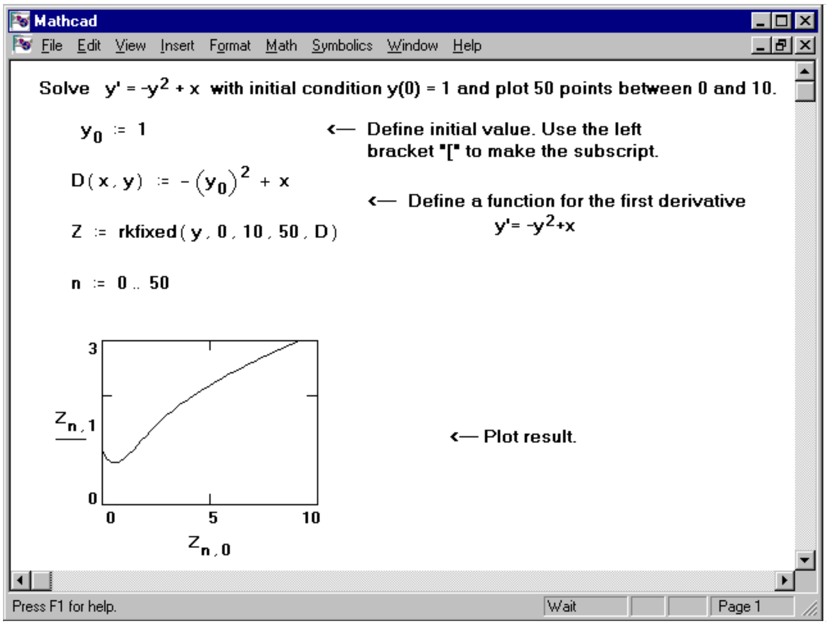
y = A vector of *n* initial values where *n* is the order of the differential equation or the size of the system of equations you're solving. For a first order differential equation like that in Figure 16-1, the vector degenerates to one point, $y(0) = y(x1)$.

x1, *x2* = The endpoints of the interval on which the solution to the differential equations will be evaluated. The initial values in **y** are the values at *x1*.

npoints = The number of points beyond the initial point at which the solution is to be approximated. This controls the number of rows $(1 + npoints)$ in the matrix returned by *rkfixed*.

D(*x*, **y**) = An *n*-element vector-valued function containing the first derivatives of the unknown functions.

The most difficult part of solving a differential equation is solving for the first derivative so you can define the function **D**(*x*, **y**). In Figure 16-1 it was easy to solve for $y'(x)$. Sometimes, however, particularly with nonlinear differential equations, it can be difficult. In such cases, you can sometimes solve for $y'(x)$ symbolically and paste it into the definition for **D**(x, **y**). To do so, use the **solve** keyword or the **Solve for Variable** command from the **Symbolics** menu as discussed in the section “Solving equations symbolically” in Chapter 17.



Solve $y' = -y^2 + x$ with initial condition y(0) = 1 and plot 50 points between 0 and 10.

$y_0 := 1$ ← Define initial value. Use the left bracket "[" to make the subscript.

$D(x, y) := -(y_0)^2 + x$ ← Define a function for the first derivative $y' = -y^2 + x$

$Z := \text{rkfixed}(y, 0, 10, 50, D)$

$n := 0 .. 50$

← Plot result.

Figure 16-2: A more complicated example involving a nonlinear differential equation.

Second order differential equations

Once you know how to solve a first order differential equation, you're most of the way to knowing how to solve higher order differential equations. We start with a second order equation. The key differences are:

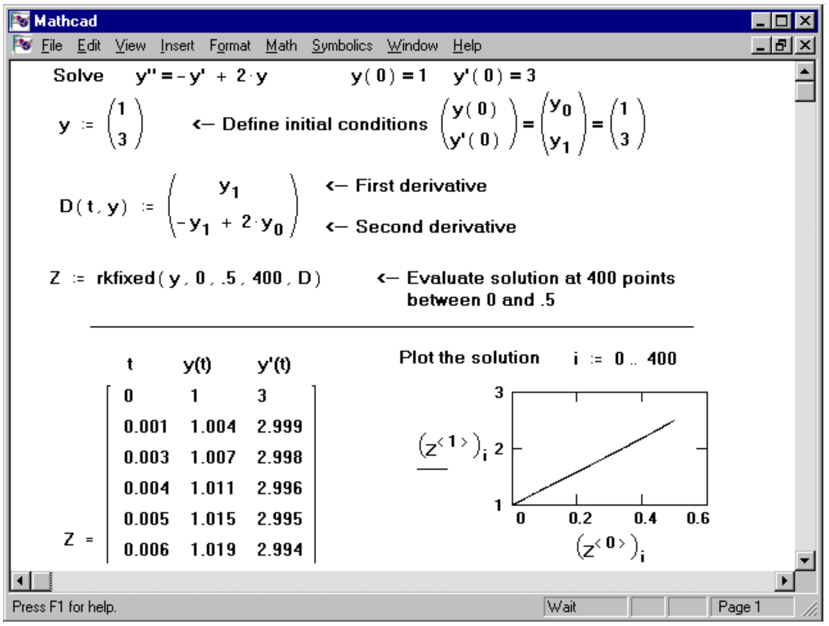
- The vector of initial values **y** now has two elements: the value of the function and its first derivative at the starting value, *x1*.
- The function **D**(t, **y**) is now a vector with two elements:

$$\mathbf{D}(t, \mathbf{y}) = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$$

- The solution matrix contains three columns: the left-hand one for the t values; the middle one for $y(t)$; and the right-hand one for $y'(t)$.

The example in Figure 16-3 shows how to solve the second order differential equation:

$$y'' = y' + 2 \cdot y$$
$$y(0) = 1 \qquad y'(0) = 3$$



Solve $y'' = -y' + 2 \cdot y \qquad y(0) = 1 \quad y'(0) = 3$

$y := \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ <— Define initial conditions $\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$D(t, y) := \begin{pmatrix} y_1 \\ -y_1 + 2 \cdot y_0 \end{pmatrix}$ <— First derivative <— Second derivative

$Z := \text{rkfixed}(y, 0, .5, 400, D)$ <— Evaluate solution at 400 points between 0 and .5

$$Z = \begin{array}{ccc} t & y(t) & y'(t) \\ 0 & 1 & 3 \\ 0.001 & 1.004 & 2.999 \\ 0.003 & 1.007 & 2.998 \\ 0.004 & 1.011 & 2.996 \\ 0.005 & 1.015 & 2.995 \\ 0.006 & 1.019 & 2.994 \end{array}$$

Plot the solution $i := 0 .. 400$

$\left(Z^{\langle 1 \rangle}\right)_i$
$\left(Z^{\langle 0 \rangle}\right)_i$
3
2
1
0
0.2
0.4
0.6

Figure 16-3: Solving a second order differential equation.

Higher order equations

The procedure for solving higher order differential equations is an extension of that used for second order differential equations. The main difference is that:

- The vector of initial values **y** now has *n* elements for specifying initial conditions of $y, y', y'', \ldots, y^{(n-1)}$.
- The function **D** is now a vector with *n* elements:

$$\mathbf{D}(t, \mathbf{y}) = \begin{bmatrix} y'(t) \\ y''(t) \\ . \\ . \\ . \\ y^{(n)}(t) \end{bmatrix}$$

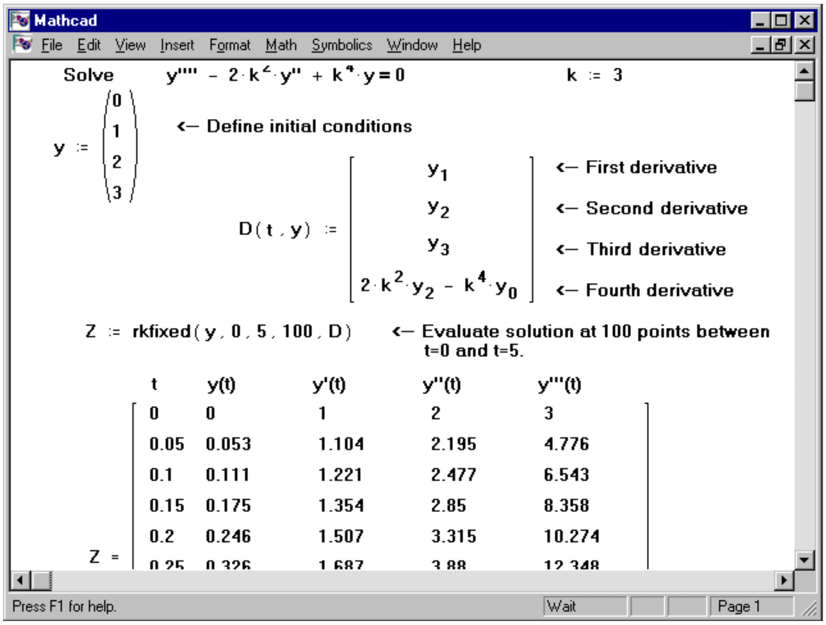
- The solution matrix contains *n* columns: the left-hand one for the *t* values and the remaining columns for values of $y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t)$.

The example in Figure 16-4 shows how to solve the fourth order differential equation:

$$y'''' - 2k^2 y'' + k^4 y = 0$$

subject to the initial conditions:

$$y(0) = 0 \qquad y'(0) = 1 \qquad y''(0) = 2 \qquad y'''(0) = 3$$



Solve $y'''' - 2\cdot k^2\cdot y'' + k^4\cdot y = 0$ $\qquad k := 3$

$y := \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ <— Define initial conditions

$$D(t, y) := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 2\cdot k^2\cdot y_2 - k^4\cdot y_0 \end{bmatrix} \begin{array}{l} \text{<— First derivative} \\ \text{<— Second derivative} \\ \text{<— Third derivative} \\ \text{<— Fourth derivative} \end{array}$$

$Z := \text{rkfixed}(y, 0, 5, 100, D)$ <— Evaluate solution at 100 points between t=0 and t=5.

Z =

t	y(t)	y'(t)	y''(t)	y'''(t)
0	0	1	2	3
0.05	0.053	1.104	2.195	4.776
0.1	0.111	1.221	2.477	6.543
0.15	0.175	1.354	2.85	8.358
0.2	0.246	1.507	3.315	10.274
0.25	0.326	1.687	3.88	12.348

Figure 16-4: Solving a higher order differential equation.

Systems of differential equations

The procedure for solving a coupled system of differential equations follows closely that for solving a higher order differential equation. In fact, you can think of solving a higher order differential equation as just a special case of solving a system of differential equations.

Systems of first order differential equations

To solve a system of first order differential equations:

- Define a vector containing the initial values of each unknown function.
- Define a vector-valued function containing the first derivatives of each of the unknown functions.
- Decide which points you want to evaluate the solutions at.
- Pass all this information into *rkfixed*.

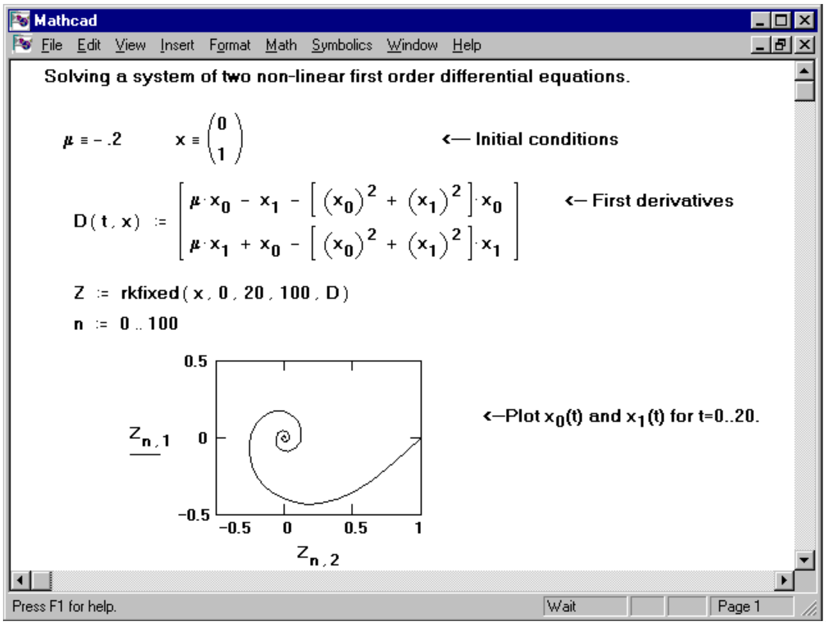
The *rkfixed* function will return a matrix whose first column contains the points at which the solutions are evaluated and whose remaining columns contain the solution functions evaluated at the corresponding point. Figure 16-5 shows an example solving the equations:

$$x'_0(t) = \mu \cdot x_0(t) - x_1(t) - (x_0(t)^2 + x_1(t)^2) \cdot x_0(t)$$

$$x'_1(t) = \mu \cdot x_1(t) - x_0(t) - (x_0(t)^2 + x_1(t)^2) \cdot x_1(t)$$

with initial conditions:

$$x_0(0) = 0 \qquad x_1(0) = 1$$



Solving a system of two non-linear first order differential equations.

$\mu \equiv -.2 \qquad x \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← Initial conditions

$$D(t,x) := \begin{bmatrix} \mu \cdot x_0 - x_1 - \left[(x_0)^2 + (x_1)^2 \right] \cdot x_0 \\ \mu \cdot x_1 + x_0 - \left[(x_0)^2 + (x_1)^2 \right] \cdot x_1 \end{bmatrix}$$ ← First derivatives

$Z := \text{rkfixed}(x, 0, 20, 100, D)$

$n := 0 .. 100$

←Plot x_0(t) and x_1(t) for t=0..20.

Figure 16-5: A system of first order linear equations.

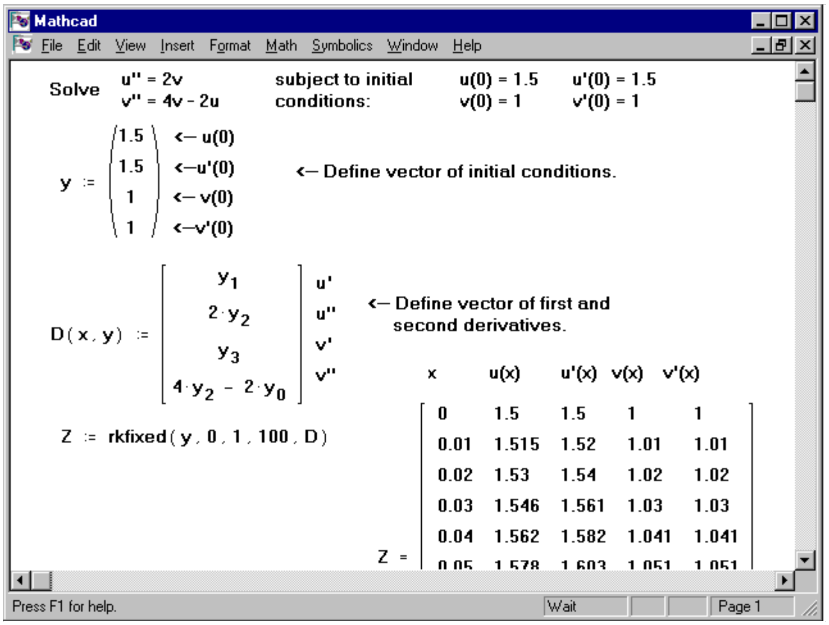
Systems of higher order differential equations

The procedure for solving a system of *n*th order differential equations is similar to the procedure for solving a system of first order differential equations. The main differences are:

- The vector of initial conditions must contain initial values for the $n - 1$ derivatives of each unknown function in addition to initial values for the functions themselves.
- The vector-valued function must contain expressions for the $n - 1$ derivatives of each unknown function in addition to the *n*th derivative.

The example in Figure 16-6 shows how to go about solving the system of second order differential equations:

$$u''(t) = 2v(t)$$
$$v''(t) = 4v(t) - 2u(t)$$



Solve $u'' = 2v$, $v'' = 4v - 2u$ subject to initial conditions: $u(0) = 1.5$, $u'(0) = 1.5$, $v(0) = 1$, $v'(0) = 1$

$$y := \begin{pmatrix} 1.5 \\ 1.5 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow u(0) \\ \leftarrow u'(0) \\ \leftarrow v(0) \\ \leftarrow v'(0) \end{matrix}$$

<— Define vector of initial conditions.

$$D(x, y) := \begin{bmatrix} y_1 \\ 2 \cdot y_2 \\ y_3 \\ 4 \cdot y_2 - 2 \cdot y_0 \end{bmatrix} \begin{matrix} u' \\ u'' \\ v' \\ v'' \end{matrix}$$

<— Define vector of first and second derivatives.

$Z := \text{rkfixed}(y, 0, 1, 100, D)$

Z =

x	u(x)	u'(x)	v(x)	v'(x)
0	1.5	1.5	1	1
0.01	1.515	1.52	1.01	1.01
0.02	1.53	1.54	1.02	1.02
0.03	1.546	1.561	1.03	1.03
0.04	1.562	1.582	1.041	1.041
0.05	1.578	1.603	1.051	1.051

Figure 16-6: A system of second order linear differential equations.

The function *rkfixed* returns a matrix in which:

- The first column contains the values at which the solutions and their derivatives are to be evaluated.
- The remaining columns contain the solutions and their derivatives evaluated at the corresponding point in the first column. The order in which the solution and its derivatives appear matches the order in which you put them into the vector of initial conditions that you passed into *rkfixed*.

Specialized differential equation solvers

Pro

The *rkfixed* function discussed thus far is a good general-purpose differential equation solver. Although it is not always the fastest method, the Runge-Kutta technique used by this function nearly always succeeds. Mathcad Professional includes several more specialized functions for solving differential equations, and there are cases in which you may want to use one of Mathcad's more specialized differential equation solvers. These cases fall into three broad categories:

- Your system of differential equations may have certain properties which are best exploited by functions other than *rkfixed*. The system may be stiff (*Stiffb*, *Stiffr*); the functions could be smooth (*Bulstoer*) or slowly varying (*Rkadapt*).
- You may have a boundary value rather than an initial value problem (*sbval* and *bvalfit*).
- You may be interested in evaluating the solution only at one point (*bulstoer*, *rkadapt*, *stiffb* and *stiffr*).

You may also want to try several methods on the same differential equation to see which one works the best. Sometimes there are subtle differences between differential equations that make one method better than another.

The following sections describe the use of the various differential equation solvers and the circumstances in which they are likely to be useful.

Smooth systems

When you know the solution is smooth, use the *Bulstoer* function instead of *rkfixed*. The *Bulstoer* function uses the Bulirsch-Stoer method rather than the Runge-Kutta method used by *rkfixed*. Under these circumstances, the solution will be slightly more accurate than that returned by *rkfixed*.

The argument list and the matrix returned by *Bulstoer* is identical to that for *rkfixed*.

Pro Bulstoer(**y**, *x1*, *x2*, *npoints*, **D**)

y =	A vector of *n* initial values.
x1, *x2* =	The endpoints of the interval on which the solution to the differential equations will be evaluated. The initial values in **y** are the values at *x1*.
npoints =	The number of points beyond the initial point at which the solution is to be approximated. This controls the number of rows ($1 + npoints$) in the matrix returned by *Bulstoer*.
D(*x*, **y**) =	An *n*-element vector-valued function containing the first derivatives of the unknown functions.

Slowly varying solutions

Given a fixed number of points, you can approximate a function more accurately if you evaluate it frequently wherever it's changing fast and infrequently wherever it's changing more slowly. If you know that the solution has this property, you may be better off using *Rkadapt*. Unlike *rkfixed* which evaluates a solution at equally spaced intervals, *Rkadapt* examines how fast the solution is changing and adapts its step size accordingly. This “adaptive step size control” enables *Rkadapt* to focus on those parts of the integration domain where the function is rapidly changing rather than wasting time integrating a function where it isn't changing all that rapidly.

Note that although *Rkadapt* will use nonuniform step sizes internally when it solves the differential equation, it will nevertheless return the solution at equally spaced points.

Rkadapt takes the same arguments as *rkfixed*. The matrix returned by *Rkadapt* is identical in form to that returned by *rkfixed*.

Pro Rkadapt(**y**, *x1*, *x2*, *npoints*, **D**)

y =	A vector of *n* initial values.
x1, *x2* =	The endpoints of the interval on which the solution to the differential equations will be evaluated. The initial values in **y** are the values at *x1*.
npoints =	The number of points beyond the initial point at which the solution is to be approximated. This controls the number of rows $(1 + npoints)$ in the matrix returned by *Rkadapt*.
D(*x*, **y**) =	An *n*-element vector-valued function containing the first derivatives of the unknown functions.

Stiff systems

A system of differential equations expressed in the form:

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}$$

is a stiff system if the matrix **A** is nearly singular. Under these conditions, the solution returned by *rkfixed* may oscillate or be unstable. When solving a stiff system, you should use one of the two differential equation solvers specifically designed for stiff systems: *Stiffb* and *Stiffr*. These use the Bulirsch-Stoer method and the Rosenbrock method, respectively, for stiff systems.

The form of the matrix returned by these functions is identical to that returned by *rkfixed*. However, *Stiffb* and *Stiffr* require an extra argument in the following section:

Pro

Stiffb(**y**, *x1*, *x2*, *npoints*, **D**, **J**)
Stiffr(**y**, *x1*, *x2*, *npoints*, **D**, **J**)

y =	A vector of *n* initial values.
x1, *x2* =	The endpoints of the interval on which the solution to the differential equations will be evaluated. The initial values in **y** are the values at *x1*.
npoints =	The number of points beyond the initial point at which the solution is to be approximated. This controls the number of rows $(1 + npoints)$ in the matrix returned by *Stiffb* or *Stiffr*.
D(*x*, **y**) =	An *n*-element vector-valued function containing the first derivatives of the unknown functions.
J(*x*, **y**) =	A function which returns the $n \times (n+1)$ matrix whose first column contains the derivatives $\partial \mathbf{D} / \partial x$ and whose remaining rows and columns form the Jacobian matrix $(\partial \mathbf{D} / \partial y_k)$ for the system of differential equations. For example, if:

$$\mathbf{D}(x, \mathbf{y}) = \begin{bmatrix} x \cdot y_1 \\ -2 \cdot y_1 \cdot y_0 \end{bmatrix} \quad \text{then} \quad \mathbf{J}(x, \mathbf{y}) = \begin{bmatrix} y_1 & 0 & x \\ 0 & -2 \cdot y_1 & -2 \cdot y_0 \end{bmatrix}$$

Evaluating only the final value

The differential equation functions discussed so far presuppose that you're interested in seeing the solution *y*(*x*) over a number of uniformly spaced *x* values in the integration interval bounded by *x1* and *x2*. There may be times, however, when all you want is the value of the solution at the endpoint, *y*(*x2*). Although the functions discussed so far will certainly give you *y*(*x2*), they also do a lot of unnecessary work returning intermediate values of *y*(*x*) in which you have no interest.

If you're only interested in the value of *y*(*x2*), use the functions listed below. Each function corresponds to one of those already discussed. The properties of each of these functions are identical to those of the corresponding function in the previous sections.

Pro

bulstoer(**y**, *x1*, *x2*, *acc*, **D**, *kmax*, *save*)
rkadapt(**y**, *x1*, *x2*, *acc*, **D**, *kmax*, *save*)
stiffb(**y**, *x1*, *x2*, *acc*, **D**, **J**, *kmax*, *save*)
stiffr(**y**, *x1*, *x2*, *acc*, **D**, **J**, *kmax*, *save*)

y =	A vector of *n* initial values.
x1, *x2* =	The endpoints of the interval on which the solution to the differential equations will be evaluated. The initial values in **y** are the values at *x1*.
acc =	Controls the accuracy of the solution. A small value of *acc* forces the algorithm to take smaller steps along the trajectory, thereby increasing the accuracy of the solution. Values of *acc* around 0.001 will generally yield accurate solutions.
D(*x*, **y**) =	An *n*-element vector-valued function containing the first derivatives of the unknown functions.
J(*x*, **y**) =	A function which returns the $n \times (n+1)$ matrix whose first column contains the derivatives $\partial\mathbf{D}/\partial x$ and whose remaining rows and columns form the Jacobian matrix ($\partial\mathbf{D}/\partial y_k$) for the system of differential equations. See page <Reference>.
kmax =	The maximum number of intermediate points at which the solution will be approximated. The value of *kmax* places an upper bound on the number of rows of the matrix returned by these functions.
save =	The smallest allowable spacing between the values at which the solutions are to be approximated. This places a lower bound on the difference between any two numbers in the first column of the matrix returned by the function.

Boundary value problems

So far, all the functions discussed in this chapter assume that you know the value taken by the solutions and their derivatives at the beginning of the interval of integration. In other words, these functions are useful for solving initial value problems.

In many cases, however, you may know the value taken by the solution at the endpoints of the interval of integration. A good example is a stretched string constrained at both ends. Problems such as this are referred to as boundary value problems. The first section discusses two-point boundary value problems: one-dimensional systems of differential equations in which the solution is a function of a single variable and the value of the solution is known at two points. The section following this discusses the more general case involving partial differential equations.

Two-point boundary value problems

The functions described so far involve finding the solution to an *n*th order differential equation when you know the value of the solution and its first $n - 1$ derivatives at the beginning of the interval of integration. This section discusses what happens if you don't have all this information about the solution at the beginning of the interval of integration but you do know something about the solution elsewhere in the interval. In particular:

- You have an *n*th order differential equation.
- You know some but not all of the values of the solution and its first $n - 1$ derivatives at the beginning of the interval of integration, *x1*.
- You know some but not all of the values of the solution and its first $n - 1$ derivatives at the end of the interval of integration, *x2*.
- Between what you know about the solution at *x1* and what you know about it at *x2*, you have *n* known values.

When this is the case, you should use *sbval* to evaluate the missing initial values at *x1*. Once you have these missing initial values, you will have an initial value problem rather than a two-point boundary value problem. You can then proceed to solve this using any of the functions discussed earlier in this chapter.

The example in Figure 16-7 shows how to use *sbval*. Note that *sbval* does not actually return a solution to a differential equation. It merely computes the initial values the solution must have in order for the solution to match the final values you specify. You must then take the initial values returned by *sbval* and solve the resulting initial value problem as discussed earlier in this chapter.

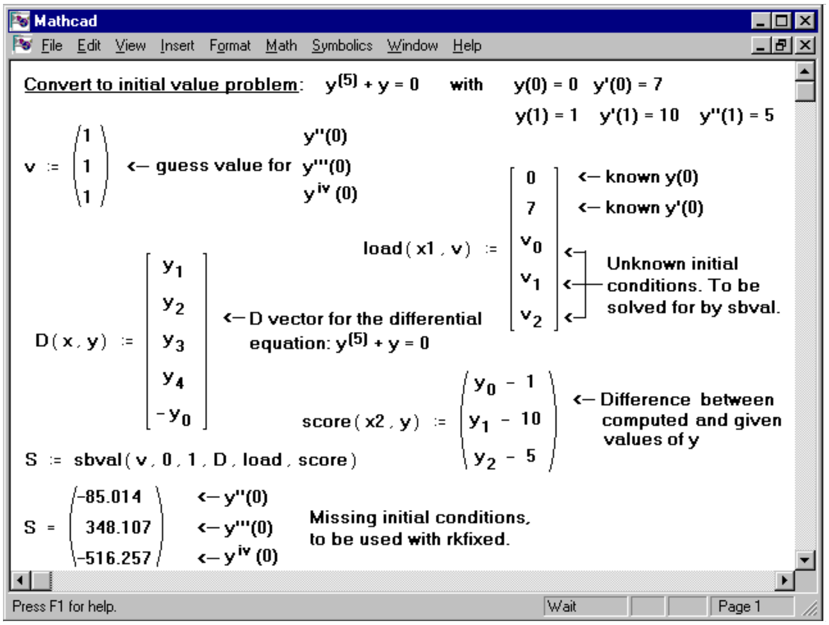
The *sbval* function returns a vector containing those initial values left unspecified at *x1*. The arguments to *sbval* are:

Pro sbval(**v**, *x1*, *x2*, **D**, **load**, **score**)

v =	Vector of guesses for initial values left unspecified at *x1*.
x1, *x2* =	The endpoints of the interval on which the solution to the differential equations will be evaluated.
D(*x*, **y**) =	An *n*-element vector-valued function containing the first derivatives of the unknown functions.
load(*x1*, **v**) =	A vector-valued function whose *n* elements correspond to the values of

the *n* unknown functions at *x1*. Some of these values will be constants specified by your initial conditions. Others will be unknown at the outset but will be found by *sbval*. If a value is unknown you should use the corresponding guess value from **v**.

score(*x2*, **y**) = A vector-valued function having the same number of elements as **v**. Each element is the difference between an initial condition at *x2*, as originally specified, and the corresponding estimate from the solution. The *score* vector measures how closely the proposed solution matches the initial conditions at *x2*. A value of 0 for any element indicates a perfect match between the corresponding initial condition and that returned by *sbval*.



Convert to initial value problem: $y^{(5)} + y = 0$ with $y(0) = 0$ $y'(0) = 7$ $y(1) = 1$ $y'(1) = 10$ $y''(1) = 5$

$$v := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{<— guess value for} \quad \begin{matrix} y''(0) \\ y'''(0) \\ y^{iv}(0) \end{matrix}$$

$$\mathrm{load}(x1, v) := \begin{bmatrix} 0 \\ 7 \\ v_0 \\ v_1 \\ v_2 \end{bmatrix} \quad \begin{matrix} \text{<— known } y(0) \\ \text{<— known } y'(0) \\ \\ \\ \\ \end{matrix}$$

<— Unknown initial conditions. To be solved for by sbval.

$$D(x, y) := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ -y_0 \end{bmatrix}$$

<— D vector for the differential equation: $y^{(5)} + y = 0$

$$\mathrm{score}(x2, y) := \begin{pmatrix} y_0 - 1 \\ y_1 - 10 \\ y_2 - 5 \end{pmatrix}$$

<— Difference between computed and given values of y

$S := \mathrm{sbval}(v, 0, 1, D, \mathrm{load}, \mathrm{score})$

$$S = \begin{pmatrix} -85.014 \\ 348.107 \\ -516.257 \end{pmatrix} \quad \begin{matrix} \text{<— } y''(0) \\ \text{<— } y'''(0) \\ \text{<— } y^{iv}(0) \end{matrix}$$

Missing initial conditions, to be used with rkfixed.

Figure 16-7: Using sbval to obtain initial values corresponding to given final values of a solution to a differential equation.

It's also possible that you don't have all the information you need to use *sbval* but you do know something about the solution and its first $n - 1$ derivatives at some intermediate value, *xf*. This is the exactly the situation contemplated by *bvalfit*.

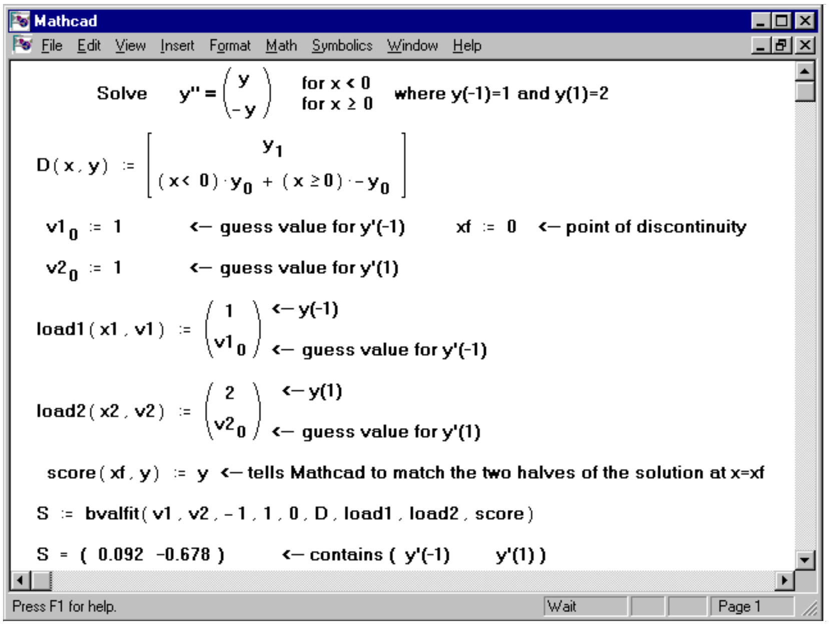
This function solves a two-point boundary value problem of this type by shooting from the endpoints and matching the trajectories of the solution and its derivatives at the intermediate point.

Pro

bvalfit(**v1**, **v2**, *x1*, *x2*, *xf*, **D**, **load1**, **load2**, **score**)

v1, **v2** =	Vector **v1** contains guesses for initial values left unspecified at *x1*. Vector **v2** contains guesses for initial values left unspecified at *x2*.
x1, *x2* =	The endpoints of the interval on which the solution to the differential equations will be evaluated.
xf =	A point between *x1* and *x2* at which the trajectories of the solutions beginning at *x1* and those beginning at *x2* are constrained to be equal.
D(*x*, **y**) =	An *n*-element vector-valued function containing the first derivatives of the unknown functions.
load1(*x1*, **v1**) =	A vector-valued function whose *n* elements correspond to the values of the *n* unknown functions at *x1*. Some of these values will be constants specified by your initial conditions. If a value is unknown you should use the corresponding guess value from **v1**.
load2(*x2*, **v2**) =	Analogous to *load1* but for values taken by the *n* unknown functions at *x2*.
score(*xf*, **y**) =	An *n* element vector valued function used to specify how you want the solutions to match at *xf*. You'll usually want to define *score*(*xf*, **y**) := **y** to make the solutions to all unknown functions match up at *xf*.

This method becomes especially useful when derivative has a discontinuity somewhere in the integration interval as the example in Figure 16-8 illustrates.



Solve $y'' = \begin{pmatrix} y \\ -y \end{pmatrix}$ for $x < 0$, for $x \ge 0$ where y(-1)=1 and y(1)=2

$D(x, y) := \begin{bmatrix} y_1 \\ (x < 0) \cdot y_0 + (x \ge 0) \cdot -y_0 \end{bmatrix}$

$v1_0 := 1$ <— guess value for y'(-1) $xf := 0$ <— point of discontinuity

$v2_0 := 1$ <— guess value for y'(1)

$load1(x1, v1) := \begin{pmatrix} 1 \\ v1_0 \end{pmatrix}$ <— y(-1), <— guess value for y'(-1)

$load2(x2, v2) := \begin{pmatrix} 2 \\ v2_0 \end{pmatrix}$ <— y(1), <— guess value for y'(1)

$score(xf, y) := y$ <— tells Mathcad to match the two halves of the solution at x=xf

$S := bvalfit(v1, v2, -1, 1, 0, D, load1, load2, score)$

$S = (\ 0.092\ \ -0.678\ )$ <— contains (y'(-1) y'(1))

Figure 16-8: Using bvalfit to match solutions in the middle of the integration interval.

Partial differential equations

A second type of boundary value problem arises when you are solving a partial differential equation. Rather than fixing the value of a solution at two points as was done in the previous section, we now fix the solution at a whole continuum of points representing some boundary.

Two partial differential equations that arise often in the analysis of physical systems are Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$$

and its homogeneous form, Laplace's equation.

Mathcad has two functions for solving these equations over a square boundary. You should use the *relax* function if you know the value taken by the unknown function $u(x, y)$ on all four sides of a square region.

If $u(x, y)$ is zero on all four sides of the square, you can use *multigrid* function instead. This function will often solve the problem faster than *relax*. Note that if the boundary condition is the same on all four sides, you can simply transform the equation to an equivalent one in which the value is zero on all four sides.

The *relax* function returns a square matrix in which:

- An element's location in the matrix corresponds to its location within the square region, and
- Its value approximates the value of the solution at that point.

This function uses the relaxation method to converge to the solution. Poisson's equation on a square domain is represented by:

$$a_{j,k}u_{j+1,k} + b_{j,k}u_{j-1,k} + c_{j,k}u_{j,k+1} + d_{j,k}u_{j,k-1} + e_{j,k}u_{j,k} = f_{j,k}$$

The arguments taken by these functions are shown below:

Pro relax(**a**, **b**, **c**, **d**, **e**, **f**, **u**, *rjac*)

a, **b**, **c**, **d**, **e** = Square matrices all of the same size containing coefficients of the above equation.

f = Square matrix containing the source term at each point in the region in which the solution is sought.

u = Square matrix containing boundary values along the edges of the region and initial guesses for the solution inside the region.

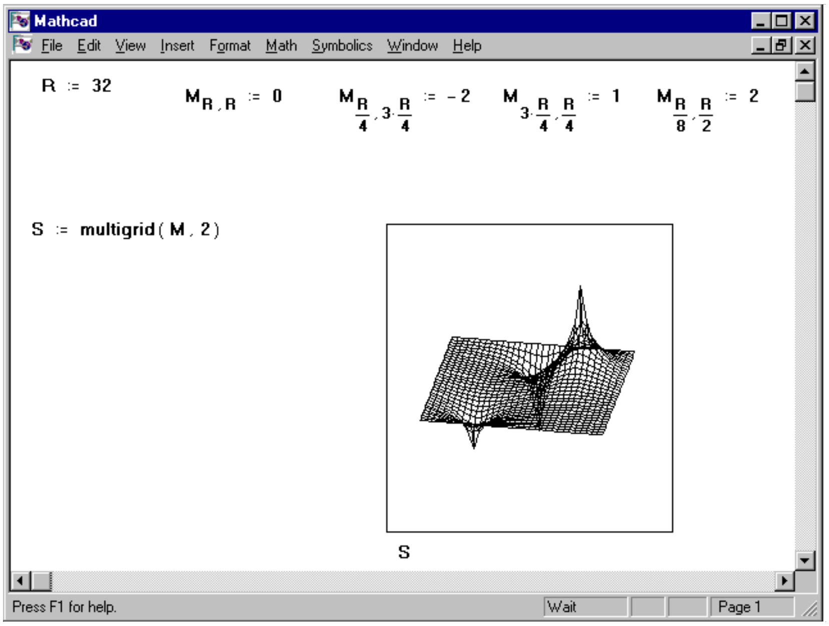
rjac = Spectral radius of the Jacobi iteration. This number between 0 and 1 controls the convergence of the relaxation algorithm. Its optimal value depends on the details of your problem.

If the boundary condition is zero on all four sides of the square integration domain, use the *multigrid* function instead. An example is shown in Figure 16-9. The same problem solved with the *relax* function instead is shown in Figure 16-10.

Pro multigrid(**M**, *ncycle*)

M = $(1 + 2^n)$ row square matrix whose elements correspond to the source term at the corresponding point in the square domain.

ncycle = The number of cycles at each level of the *multigrid* iteration. A value of 2 will generally give a good approximation of the solution.

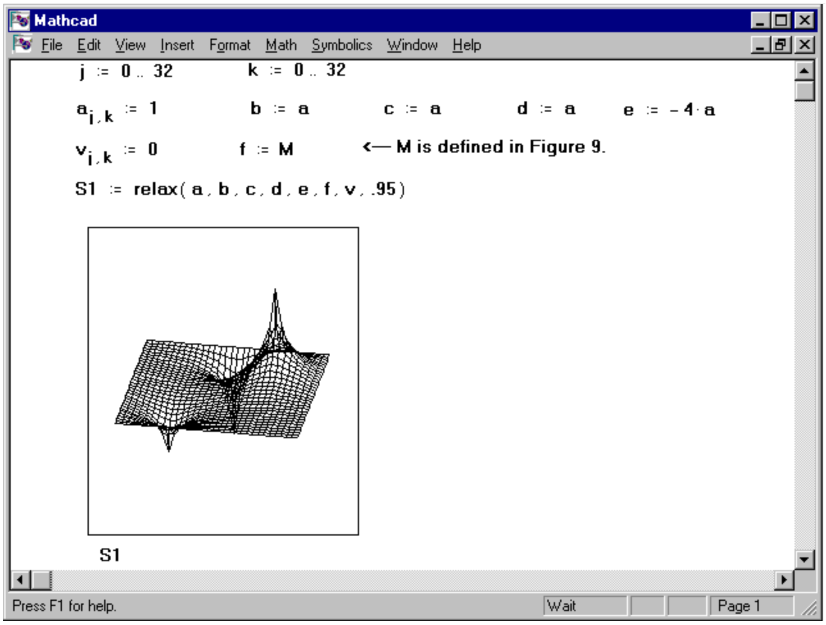


$R := 32$ $M_{R,R} := 0$ $M_{\frac{R}{4},3\cdot\frac{R}{4}} := -2$ $M_{3\cdot\frac{R}{4},\frac{R}{4}} := 1$ $M_{\frac{R}{8},\frac{R}{2}} := 2$

$S := \mathrm{multigrid}(M, 2)$

S

Figure 16-9: Using multigrid to solve a Poisson's equation in a square domain.



$j := 0 .. 32$ $k := 0 .. 32$

$a_{j,k} := 1$ $b := a$ $c := a$ $d := a$ $e := -4\cdot a$

$v_{j,k} := 0$ $f := M$ ←— M is defined in Figure 9.

$S1 := \mathrm{relax}(a, b, c, d, e, f, v, .95)$

S1

Figure 16-10: Using relax to solve the same problem as that shown in Figure 16-9.